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Advanced Linear Algebra (MA 409)
Problem Sheet - 16

Determinants of order n

1. Label the following statements as true or false.

   (a) The function $\det : M_{n\times n}(F) \to F$ is a linear transformation.

   (b) The determinant of a square matrix can be evaluated by cofactor expansion along any row.

   (c) If two rows of a square matrix A are identical, then $\det(A) = 0$.

   (d) If B is a matrix obtained from a square matrix A by interchanging any two rows, then $\det(B) = -\det(A)$.

   (e) If B is a matrix obtained from a square matrix A by multiplying a row of A by a scalar, then $\det(B) = \det(A)$.

   (f) If B is a matrix obtained from a square matrix A by adding k times row i to row j, then $\det(B) = k\det(A)$.

   (g) If $A \in M_{n\times n}(F)$ has rank n, then $\det(A) = 0$.

   (h) The determinant of an upper triangular matrix equals the product of its diagonal entries.

2. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 3a_1 & 3a_2 & 3a_3 \\ 3b_1 & 3b_2 & 3b_3 \\ 3c_1 & 3c_2 & 3c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

3. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} 2a_1 & 2a_2 & 2a_3 \\ 3b_1+5c_1 & 3b_2+5c_2 & 3b_3+5c_3 \\ 7c_1 & 7c_2 & 7c_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

4. Find the value of k that satisfies the following equation:

$$\det\begin{pmatrix} b_1+c_1 & b_2+c_2 & b_3+c_3 \\ a_1+c_1 & a_2+c_2 & a_3+c_3 \\ a_1+b_1 & a_2+b_2 & a_3+b_3 \end{pmatrix} = k\det\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}.$$

In Exercises 5-12, evaluate the determinant of the given matrix by cofactor expansion along the indicated row.

5. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$
along the first row

6. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$
along the first row

7. $\begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & -3 \\ 2 & 3 & 0 \end{pmatrix}$
along the second row

8. $\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 5 \\ -1 & 3 & 0 \end{pmatrix}$
along the third row

9. $\begin{pmatrix} 0 & 1+i & 2 \\ -2i & 0 & 1-i \\ 3 & 4i & 0 \end{pmatrix}$
along the third row

10. $\begin{pmatrix} i & 2+i & 0 \\ -1 & 3 & 2i \\ 0 & -1 & 1-i \end{pmatrix}$
along the second row

11. $\begin{pmatrix} 0 & 2 & 1 & 3 \\ 1 & 0 & -2 & 2 \\ 3 & -1 & 0 & 1 \\ -1 & 1 & 2 & 0 \end{pmatrix}$
along the fourth row

12. $\begin{pmatrix} 1 & -1 & 2 & -1 \\ -3 & 4 & 1 & -1 \\ 2 & -5 & -3 & 8 \\ -2 & 6 & -4 & 1 \end{pmatrix}$
along the fourth row

In Exercises 13-22, evaluate the determinant of the given matrix by any legitimate method.

13. $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

14. $\begin{pmatrix} 2 & 3 & 4 \\ 5 & 6 & 0 \\ 7 & 0 & 0 \end{pmatrix}$

15. $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$

16. $\begin{pmatrix} -1 & 3 & 2 \\ 4 & -8 & 1 \\ 2 & 2 & 5 \end{pmatrix}$

17. $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 2 & -5 \\ 6 & -4 & 3 \end{pmatrix}$

18. $\begin{pmatrix} 1 & -2 & 3 \\ -1 & 2 & -5 \\ 3 & -1 & 2 \end{pmatrix}$

19. $\begin{pmatrix} i & 2 & -1 \\ 3 & 1+i & 2 \\ -2i & 1 & 4-i \end{pmatrix}$

20. $\begin{pmatrix} -1 & 2+i & 3 \\ 1-i & i & 1 \\ 3i & 2 & -1+i \end{pmatrix}$

21. $\begin{pmatrix} 1 & 0 & -2 & 3 \\ -3 & 1 & 1 & 2 \\ 0 & 4 & -1 & 1 \\ 2 & 3 & 0 & 1 \end{pmatrix}$

22. $\begin{pmatrix} 1 & -2 & 3 & -12 \\ -5 & 12 & -14 & 19 \\ -9 & 22 & -20 & 31 \\ -4 & 9 & -14 & 15 \end{pmatrix}$

23. Prove that the determinant of an upper triangular matrix is the product of its diagonal entries.

24. Prove that if E is an elementary matrix, then $\det(E^t) = \det(E)$.

25. Prove that $\det(kA) = k^n \det(A)$ for any $A \in M_{n\times n}(F)$.

26. Let $A \in M_{n\times n}(F)$. Under what conditions is $\det(-A) = \det(A)$?

27. Prove that if $A \in M_{n\times n}(F)$ has two identical columns, then $\det(A) = 0$.

28. Compute $\det(E_i)$ if E_i is an elementary matrix of type i.

29. Let the rows of $A \in M_{n\times n}(F)$ be $a_1, a_2, \ldots, a_n$, and let B be the matrix in which the rows are $a_n, a_{n-1}, \ldots, a_1$. Calculate $\det(B)$ in terms of $\det(A)$.

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